Application of Gauss Theorem

Objectives

After going through this lesson, the learners will be able to:

- Recall Gauss Law and it's the physical significance
- Apply Gauss Law to find the electric field due to charge distribution by
  - Infinitely long straight wire
  - Uniformly charged infinite plane and
  - Uniformly charged thin spherical shell (field inside and outside)
- Plot graphs to represent the electric field around charge distributions

Content Outline

- Unit syllabus
- Module wise distribution of unit syllabus
- Words you must know
- Introduction
- Application of Gauss Law - field due to
  - Collection of charges
  - Infinitely long straight wire
  - Uniformly charged infinite plane
  - Uniformly charged thin spherical shell (field inside and outside)
- Graphs
- Summary

Unit Syllabus

Chapter-1: Electric Charges and Fields

Electric Charges; Conservation of charge, Coulomb's law-force between two point charges, forces between multiple charges; superposition principle and continuous charge distribution. Electric field; electric field due to a point charge, electric field lines, electric dipole, electric field due to a dipole, torque on a dipole in uniform electric field.

Electric flux, statement of Gauss's theorem and its applications to find field due to infinitely long straight wire, uniformly charged infinite plane sheet and uniformly charged thin spherical shell (field inside and outside).

Chapter-2: Electrostatic Potential and Capacitance

Electric potential, potential difference, electric potential due to a point charge, a dipole and system of charges; equipotential surfaces, electrical potential energy of a system of two point charges and of electric dipole in an electrostatic field.

Conductors and insulators, free charges and bound charges inside a conductor. Dielectrics and electric polarization, capacitors and capacitance, combination of capacitors in series and in parallel, capacitance of a parallel plate capacitor with and without dielectric medium between the plates, energy stored in a capacitor.

Module Wise Distribution Of Unit Syllabus

The above unit is divided into 11 modules for better understanding.

Module 1	● Electric charge ● Properties of charge ● Coulombs' law ● Characteristics of coulomb force ● Constant of proportionality and the intervening medium ● Examples
Module 2	● Forces between multiple charges ● Principle of superposition ● Continuous distribution of charges ● numerical
Module 3	● Electric field E ● Importance of field E and ways of describing field ● Superposition of electric field ● Examples
Module 4	● Electric dipole ● Electric field of a dipole ● Charges in external field

	• Dipole in external field Uniform and non-uniform
Module 5	• Electric flux, • Flux density • Gauss theorem • Application of gauss theorem to find electric field For a distribution of charges • Numerical
Module 6	• Application of gauss theorem Field due to field infinitely long straight wire • Uniformly charged infinite plane • Uniformly charged thin spherical shell (field inside and outside) • Graphs
Module 7	• Electric potential, • Potential difference, • Electric potential due to a point charge, a dipole and system of charges; • Equipotential surfaces, • Electrical potential energy of a system of two point charges and of electric dipole in an electrostatic field. • Numerical
Module 8	• Conductors and insulators, • Free charges and bound charges inside a conductor. • Dielectrics and electric polarization
Module 9	• Capacitors and Capacitance, • Combination of capacitors in series and in parallel • Redistribution of charges , common potential • numerical
Module 10	• Capacitance of a parallel plate capacitor with and without dielectric medium between the plates

	• Energy stored in a capacitor
Module 11	• Typical problems on capacitors

Words You Must Know

Let us recollect the words we have been using in our study of this physics course.

- **Electric Charge:** Electric charge is an intrinsic characteristic of many of the fundamental particles of matter that gives rise to all electric and magnetic forces and interactions. There are two kinds of charges, positive and negative.
- **Conductors:** Some substances readily allow passage of electricity through them, others do not. Those which allow electricity to pass through them easily are called *conductors*. They have electric charges (electrons) that are comparatively free to move inside the material. Metals, humans, animal bodies and earth are all conductors of electricity.
- **Insulators:** Most of the non-metals, like glass, porcelain, plastic, nylon, wood, offer high opposition to the passage of electricity through them. They are called *insulators.*
- **Point Charge:** When the linear size of charged bodies is much smaller than the distance separating them, the size may be ignored and the charge bodies can then be treated as *point charges*.
- **Conduction:** Transfer of electrons from one body to another, it also refers to flow of charged electrons in metals and ions in electrolytes and gases.
- **Induction:** The temporary separation of charges in a body due to a charged body in the vicinity. The effect lasts as long as the charged body is held close to the body in which induction is taking place.
- **Quantization of charges:** Charge exists as an integral multiple of basic electronic charge. Charge on an electron is 1.6×10^{-19} C.
- **Electroscope:** A device to detect charge, to find the relative magnitude of charge on two charged bodies. A suitably charged electroscope can be used to find the nature of charge on a charged body.
- **Coulomb:** S.I unit of charge defined in terms of 1 ampere current flowing in a wire to be due to 1 coulomb of charge flowing in 1 s.

  1 coulomb = collective charge of 6×10^{18} electrons
- **Conservation of charge:** Charge can neither be created nor destroyed in an isolated system; it (electrons) only transfers from one body to another.

- **Coulomb’s Force:** It is the electrostatic force of interaction between the two point charges.
- **Coulomb's law:** A mathematical expression based on coulomb's law to show the magnitude as well as direction of mutual electrostatic force between two or more charges.

$$F = K\frac{q_1 \times q_2}{r^2}$$

Vector form of Coulomb's law: A mathematical expression based on coulomb's law to show the magnitude as well as direction of mutual electrostatic force between two or more charges. Force between charges q_1 and q_2, F_{12} is the force on 1 due to 2, depending upon the nature of the charges (both positive, both negative or q_1 positive and q_2 negative or q_2 positive and q_1 negative)

$\hat{}_{12}$ Vector shows the direction of the force

$$F_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2_{12}}\hat{}_{12}$$

- **Laws of vector addition**

  Triangle law of vector addition: If two vectors are represented by two sides of a triangle in order, then the third side represents the resultant of the two vectors.

  Parallelogram law of vector addition: If two vectors are represented in magnitude and direction by adjacent sides of a parallelogram then the resultant of the vectors is given by the diagonal passing through their common point.

  - Also resultant of vectors **P** and **Q** acting at angle of θ is given by

  $$R = \sqrt{P^2 + Q^2 + 2PQcos\theta}$$

  Polygon law of vector addition: Multiples vectors may be added by placing them in order of a multi sided polygon, the resultant is given by the closing side taken in opposite order.

- **Linear charge density:** The *linear charge density, λ* is defined as the charge per unit length.
- **Surface charge density:** The surface *charge density* σ is defined as the charge per unit surface area.

  The surface charge density σ at the area element Δs is given by $\sigma = \frac{\Delta Q}{\Delta s}$.
- **Volume charge density:** The volume ch*arge density* ρ is defined as the charge per unit volume.

- **Superposition Principle:** For an assembly of charges q_1, q_2, q_3, ..., the force on any charge, say q_1, is the vector sum of the force on q_1 due to q_2, the force on q_1 due to q_3, and so on. For each pair, the force is given by Coulomb's law for two point charges.
- **Electric field lines:** An electric field line is a curve drawn in such a way that the tangent at each point on the curve gives the direction of the electric field at that point.
- **Area vector:** The area element vector ΔS at a point on a closed surface equals $\Delta S\,\hat{n}$ where ΔS is the magnitude of the area element and $\hat{n}$ is a unit vector in the direction of outward normal at that point..
- Electric field the space around a charge where its influence may be experienced by other charged bodies. The field strength at appoint in the field is given by E = electrostatic force per unit charge; unit is NC^{-1}
- Electric field line electric field lines in an electric field which trace the path of a unit positive charge
- Electric flux electric field lines crossing an area
- Electric flux density field lines crossing a unit area held perpendicular to the field lines represented by φ unit weber

$$\varphi = E.\Delta s$$

- **Gaussian surface:** The closed surface that we need to choose for applying Gauss's law to a particular charge distribution is called the Gaussian surface.
- **Gauss's Theorem/Law:** The flux of the electric field through any closed surface S is $1/\varepsilon_0$ times the total charge enclosed by that surface

Introduction

You will recall, we had earlier described the electric field strength in terms of electrostatic force per unit charge; you also described it in terms of number of field lines per unit area. Gauss's law is a mathematical way of finding the strength of electric fields around charges.

In module 5 we studied electric flux, we related it to water flux through a pipe of uniform cross-section.

We defined electric flux density as **electric flux per unit area vector**. This is another way of expressing electric field strength at locations around point charge, collection of charges and charge distribution.

We had described the area vector as a unit vector in the direction of outward normal to a point on the surface and Gauss law.

Now we know that Gauss law can be expressed as-

Electric flux through a closed surface $S = q/\varepsilon_0$, where q = total or net charge enclosed by S.

We can also recall that Gauss's law is applicable only under following two conditions:

- The electric field at every point on the surface is either perpendicular
- Magnitude of the electric field at every point where it is perpendicular to the surface has a constant value (say *E*). Here *S* is the area where electric field is perpendicular to the surface

The choice of Gaussian surface is ours.

As Gauss's law does not provide expression for electric fields but provides only for its flux through a closed surface.

To calculate *E,* we choose an imaginary closed surface (called Gaussian surface) in which Gauss law can be applied easily. Let us discuss a few simple cases.

Application of Gauss Law

(A) Field Due to a Collection of Charges

The figure shows a collection of charges, if we need to find the electric field due to them collectively at a position away from it.

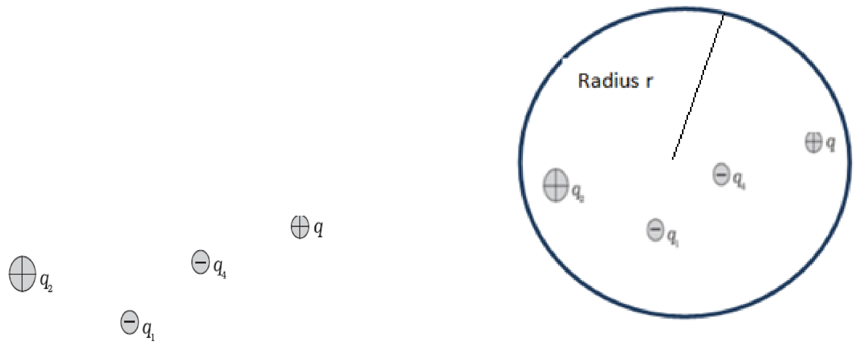
Our choices would be:

- Calculate vector sum of E at that point due to individual charges
- Use Gauss's law

To use Gauss's law which states:

The flux of electric field through any closed surface *S* is $1/\varepsilon_0$ times the total charge enclosed by *S*. The law is especially useful in determining electric field E, when the source distribution has simple symmetry.

Now imagine any surface around the charges:



Charge enclosed in a Gaussian surface of spherical symmetry

Now write the statement $E.\Delta S$ = sum of charges enclosed in the surface $/\varepsilon_0$

Notice here the assumption is:

We selected a spherical symmetry around the charges, a sphere of radius r

The figure is three-dimensional.

The charges are placed in vacuum so permittivity is ε_0.

The total charge is $q_1 + q_2 + q_4 + q$ all are not positive, therefore we will add them algebraically.

Say is:

$$q_1 = -\ 5\mu C$$

$$q_2 = +\ 6\ \mu C$$

$$q_4 = \ -\ 7\mu C$$

$$q\ =\ +\ 8\ \mu C$$

So the total charge enclosed is + 2 μC

Flux = + 2 $\mu C/\varepsilon_0$

$$E \times 4\pi r^2 = +2\ \mu C/\varepsilon_0$$

From the above we can calculate E at a distance of r from the center of the distribution.

Points we must bear in mind while using the above method for calculation of electric field:

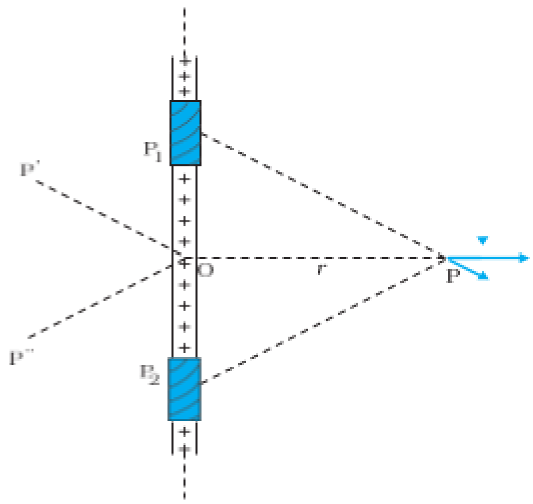
- It cannot be used for a dipole because the net charge will be zero
- If any of the charges are excluded from being inside the Gaussian surface then they must not be included in the calculation for total charge
- What if the Gaussian surface selected contains a charge ?Then it should not be a choice of surface.
- If the collection of charges within a chosen surface are imbalanced in terms of distribution about the center of Gaussian surface or magnitude, choice of surface may

be changed. **The shape of the surface is not important, but in such a case the surface area would be calculated by using calculus.**

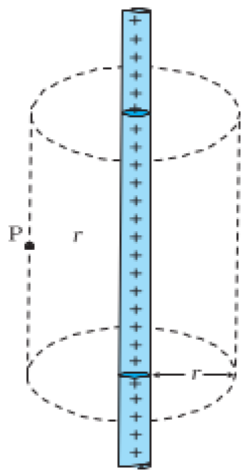
(B) Field Due To An Infinitely Long Straight Uniformly Charged Wire

Consider an infinitely long thin straight wire with uniform linear charge densityλ.

By infinitely long we mean, much longer than the linear spread of electric field to be studied. The wire is obviously an axis of symmetry.



(a)



(b)

(a) Electric field due to an infinitely long thin straight wire is radial,

(b) The Gaussian surface for a long thin wire of uniform linear charge density.

Suppose we take the radial vector from O to P and rotate it around the wire. The points P, P', P" so obtained are completely equivalent with respect to the charged wire. This implies that the electric field must have the same magnitude at these points. The direction of the electric field at every point must be radial (outward if q> 0, Inward if q< 0). This is clear from Fig. (a) Consider a pair of line elements P_1 and P_2 at the wire, as shown. The electric fields produced by the two elements of the pair when summed up give a resultant electric field which is radial (the components normal to the radial vector cancel). This is true for any such pair and hence the total field at any point P is radial. Finally, since the wire is infinite, the electric field does not depend on the position of P along the length of the wire. In short, the electric field is everywhere radial in the plane cutting the wire normally, and its magnitude depends only on the radial distance r.

To calculate the field, imagine a cylindrical Gaussian surface, as shown in the Fig. (b).

Since the field is radial everywhere, flux through the two ends of the cylindrical Gaussian surface is zero.

At the cylindrical part of the surface, E is normal to the surface at every point, and its magnitude is constant, since it depends only on r.

The surface area of the curved part is $2\pi rl$, where 'l' is the length of the cylinder.

Flux through the Gaussian surface:

= flux through the curved cylindrical part of the surface

$= E \times 2\pi rl$

The surface includes charge equal to l. Gauss's law then gives:

$$E \times 2\pi rl = \lambda l / \varepsilon_0$$

$$i.e.\ E = \frac{\lambda}{2\pi r \varepsilon_0}$$

Vectorially, **E** at any point is given by-

$$E = \frac{\lambda}{2\pi \varepsilon_0 r} \hat{n}$$

Where, $\hat{n}$ is the radial unit vector in the plane normal to the wire passing through the point,

E is directed outward if λ is positive and inward if λ is negative.

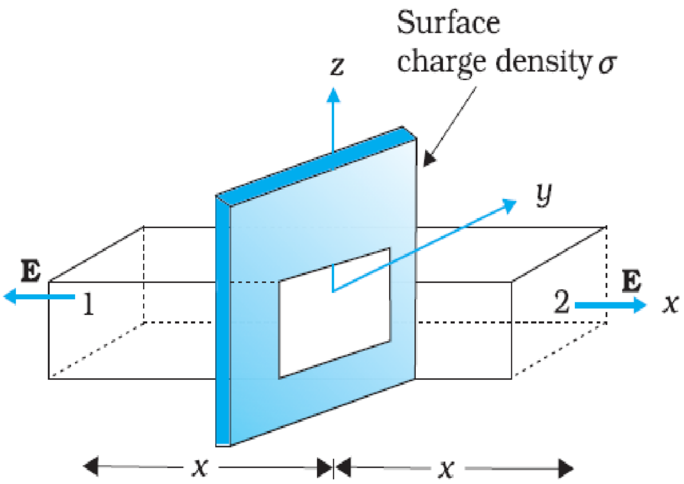
Also note that though only the charge enclosed by the surface (l) was included above, the electric field E is due to the charge on the entire wire because it is calculated in terms of linear charge density.

Further, the assumption that the wire is infinitely long is crucial. Without this assumption, we cannot take E to be normal to the curved part of the cylindrical Gaussian surface. However, the above formula is approximately true for electric fields around the central portions of a long wire, where the end effects may be ignored.

(C) Field Due To A Uniformly Charged Infinite Plane Sheet

Let σ be the uniform surface charge density of an infinite plane sheet as shown in the figure below. We take the x-axis normal to the given plane. By symmetry, the electric field will not depend on y and *z* coordinates and its direction at every point must be parallel to the x-direction.

We can take the Gaussian surface to be a rectangular parallelepiped of cross-sectional area A, as shown. (A cylindrical surface will also do.) As seen from the figure,



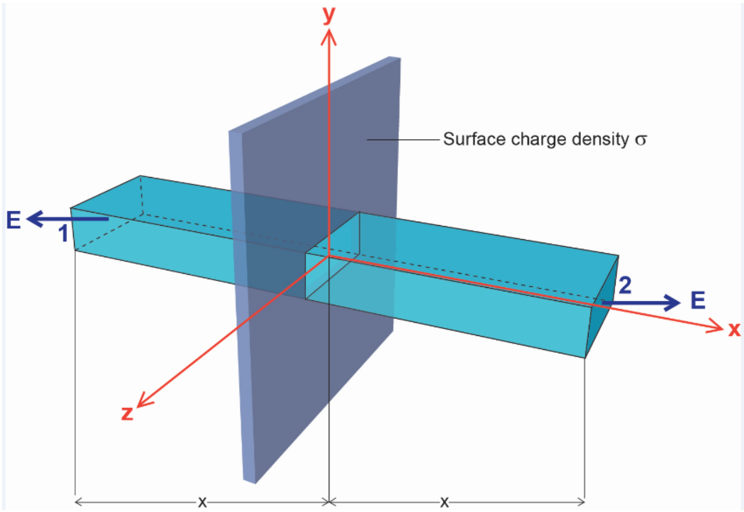
Gaussian surfaces for a uniformly charged infinite plane sheet

This figure is from the NCERT book

The error in the fig the parallelepiped should be across the central rectangle, because it represents the flux emitted from the surface. The distance x is from the sheet to faces 1 and 2 of the parallelepiped the sheet is in y-z plane and edge of the parallelepiped is along x direction Redraw and see it

Only the two faces 1 and 2 will contribute to the flux; electric field lines are parallel to the other faces and they, therefore, do not contribute to the total flux emerging out of the charged sheet.

The correct figure is:



Gaussian surfaces for a uniformly charged infinite plane sheet

The unit vector normal to surface 1 is in -*x* direction while the unit vector normal to surface 2 is in the +*x* direction.

Therefore, flux $E.S$ through both the surfaces are equal and add up.

Therefore the net flux through the Gaussian surface is $2EA$. The charge enclosed by the closed surface is σA.

Therefore by Gauss's law-

$$2E\,A = \sigma A/\varepsilon_0$$

Or, $$\mathbf{E} = \boldsymbol{\sigma}/2\boldsymbol{\varepsilon}_0$$

Vectorially,

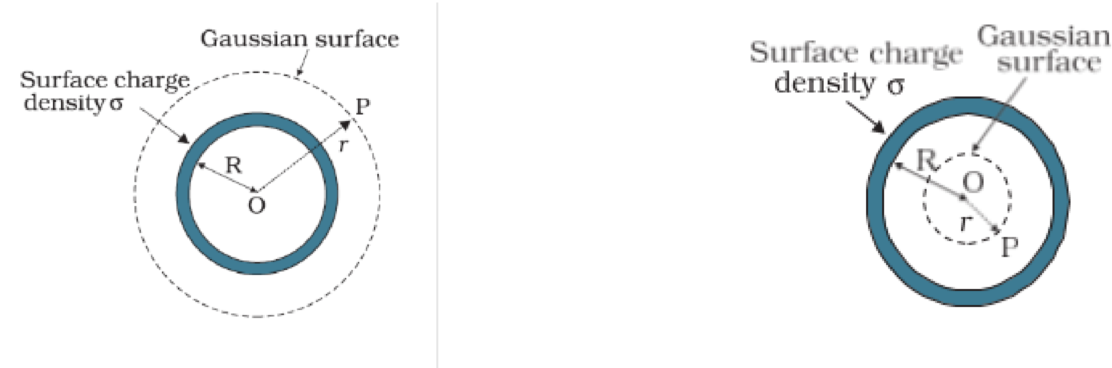
$E = \frac{\sigma}{2\epsilon 0}\hat{n}$, where $\hat{n}$ is a unit vector normal to the plane and going away from it.

E is directed away from the plate if σ is positive and toward the plate if σ is negative. Note that the above application of the Gauss' law has brought out an additional fact: *E* is independent of position of the point also.

For a finite large planar sheet, the above formula is approximately true in the middle regions of the planar sheet, away from the ends.

(D) Field Due To A Uniformly Charged Thin Spherical Shell

Let σ be the uniform surface charge density of a thin spherical shell of radius *R* (Fig. 3). The situation has an obvious spherical symmetry. The field at any point P, outside or inside, can depend only on r (the radial distance from the center of the shell to the point) and must be radial (i.e., along the radius vector)



(a) $r > R$ (b) $r < R$

Figure shows Gaussian surface

i) Field outside of the shell: Consider a point P outside the shell with radius vector r. To calculate E at point P, we take the Gaussian surface to be a sphere of radius r and with centre O, passing through P as shown in Fig 3 (a). The electric field at each point of the Gaussian surface, therefore, has the same magnitude *E* and is along the radius vector at each point. Thus, E and ΔS at every point are parallel and the flux through each element is *E* ΔS. Summing over all ΔS, the flux through the Gaussian surface is $E\times 4\pi r^2$. The charge enclosed is $\frac{}{_0}\times 4\pi R^2$. By Gauss's law

$$E \times 4\pi r^2 = \frac{\sigma}{\epsilon_0} 4\pi R^2$$

Or.

$$E = \frac{\sigma R^2}{\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r^2}$$

Where $q = 4\pi R^2\sigma$ is the total charge on the spherical shell vectorially

$$\vec{E} = \frac{q}{4\pi\ _0 r^2}\hat{r}$$

This electric field is directed outward if q>0 and inward if q<0. This, however, is exactly the field produced by a charge q placed at the center O. Thus for points outside the shell, the field due to a uniformly charged shell is as if the entire charge of the shell is concentrated at its center.

ii) Field inside the shell: In Fig. (b), the point P is inside the shell. The Gaussian surface is again a sphere through P centered at O.

The flux through the Gaussian surface, calculated as before, is $E \times 4\pi r^2$. However, in this case, the Gaussian surface encloses no charge.

Gauss's law then gives

$$E \times 4\pi r^2 = 0$$

i.e.,

$$E = 0 \quad (r < R)$$

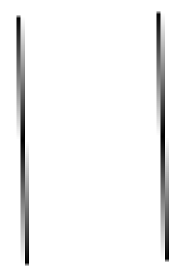
That is, the field due to a uniformly charged thin shell is zero at all points inside the shell. This important result is a direct consequence of Gauss's law which follows from Coulomb's law. The experimental verification of the result confirms the $1/r^2$ dependence in Coulomb's law.

Example:

Discuss the electric field intensity due to two thin infinite parallel charged sheets.

Solution:

Let A and B be two thin infinite plane charged sheets held parallel to each other, as shown in the figure.



A B

I σ_1 II σ_2 III

Suppose

σ_1 = Uniform surface density of charge on A

σ_2 = Uniform surface density of charge on B

If E_1, E_2 are electric field intensities at a point due to charged plane sheets A and B respectively, then

$E_1 = \frac{\sigma_1}{2\epsilon_0}$ and $E_2 = \frac{\sigma_2}{2\epsilon_0}$

The arrangement shows **three regions I, II and III.**

We apply the superposition principle to calculate the net field intensity in the three regions. **As a matter of convention, a field pointing from left to right is taken as positive and the one pointing from right to left is taken as negative.**

We assume that $\sigma_1 > \sigma_2 > 0$

In region I, $E_I = -E_1 - E_2$

$E_I = -\frac{\sigma_1}{2\epsilon_0} - \frac{\sigma_2}{2\epsilon_0} = \frac{1}{2\epsilon_0}(\sigma_1 + \sigma_2)$ directed towards the left.

Similarly, in region II

$E_{II} = E_1 - E_2 = \frac{\sigma_1}{2\epsilon_0} - \frac{\sigma_2}{2\epsilon_0} = \frac{1}{2\epsilon_0}(\sigma_1 - \sigma_2)$ directed towards the right.

And in region III

$E_{III} = E_1 + E_2 = \frac{\sigma_1}{2\epsilon_0} + \frac{\sigma_2}{2\epsilon_0} = \frac{1}{2\epsilon_0}(\sigma_1 + \sigma_2)$ directed towards right

SPECIAL CASES

Suppose $\sigma_1 = \sigma$ and $\sigma_2 = -\sigma$ i.e. **two thin infinite plane sheets with equal and opposite uniform surface densities of charge are held parallel to each other**

$$\mathbf{E_I = 0}$$

$$\mathbf{E_{III} = 0}$$

$$\mathbf{E_{II} = \frac{2\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0} = constant}$$

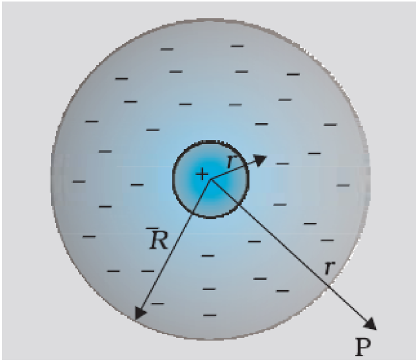
Thus **field intensity in between such sheets having equal and opposite uniform surface densities of charge becomes constant i.e., a uniform electric field is produced in between two such sheets,** Also, E does not depend upon the distance between the thin sheets.

This is how uniform electric fields are produced in practice.

Example:

An early model for an atom considered it to have a positively charged point nucleus of charge *Ze*, surrounded by a uniform density of negative charge up to a radius *R*. The atom as a whole is neutral. For this model, what is the electric field at a distance *r* from the nucleus?

Solution:



The charge distribution for this model of the atom is as shown in Fig. The total negative charge in the uniform spherical charge distribution of radius *R* must be –*Z e*, since the atom (nucleus of charge *Z e* + negative charge) is neutral.

This immediately gives us the negative charge density ρ, since we must have:

$$\frac{4\pi R^3}{3}\rho = 0 - Ze \quad or \quad \rho = -\frac{3Ze}{4\pi R^3}$$

To find the electric field E(r) at a point P which is a distance *r* away from the nucleus, we use Gauss's law. Because of the spherical symmetry of the charge distribution, the magnitude of the electric field E(r) depends only on the radial distance, no matter what the direction of r. Its direction is along (or opposite to) the radius vector r from the origin to the point P.

The obvious Gaussian surface is a spherical surface centred at the nucleus. We consider two situations,

namely, $r < R$ and $r > R$.

$$\phi = E(r)\times 4\pi r^2$$

Where *E* (*r*) is the magnitude of the electric field at *r*. This is because the field at any point on the spherical Gaussian surface has the same direction as the normal to the surface there, and has the same magnitude at all points on the surface.

The charge *q* enclosed by the Gaussian surface is the positive nuclear charge and the negative charge within the sphere of radius *r i.e.*

$$q = Ze + \frac{4\pi r^3}{3}\rho$$

Substituting for the charge density ρ obtained earlier, we have:

$$q = Ze - Ze\frac{r^3}{R^3}$$

Gauss's law then gives:

$$E(r) = \frac{Ze}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{R^3}\right); \qquad r < R$$

The electric field is directed radially outward.

i) $r > R$: In this case, the total charge enclosed by the Gaussian spherical surface is zero since the atom is neutral. Thus, from Gauss's law:

$$E(r) \times 4\pi r^2 = 0 \quad or \quad E(r) = 0; \quad r > R$$

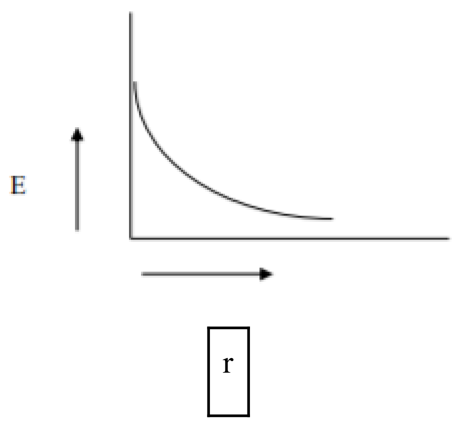
ii) At r = R, both cases give the same result: E = 0.

Graphs To Show Variation of Electric Field Intensity with Distance

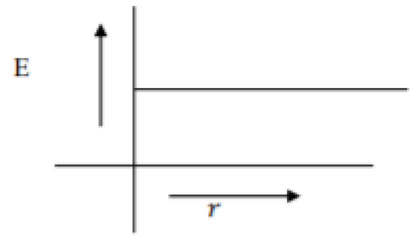
- Variation of Electric field due to **infinite long straight uniformly charged wire with position vector r.**

$$E = \frac{\lambda}{2\pi\varepsilon 0 r}$$

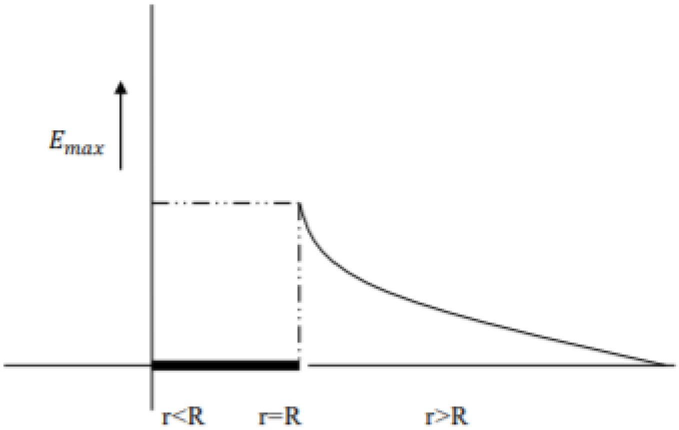
$$y = K_1/r$$



- Variation of Electric field due to uniformly **charged infinite plane sheet with position vector r**



● Variation of Electric field due to a **uniformly charged thin spherical shell having radius R with position vector r.**



Summary

Gauss's law: The flux of electric field through any closed surface *S* is$1/\varepsilon_0$ times the total charge enclosed by *S*. The law is especially useful in determining electric field E, when the source distribution has simple symmetry:

● Electric field due to a thin infinitely long straight wire of uniform linear charge density λ

$$E = \frac{\lambda}{2\pi\varepsilon 0 r}\hat{n}$$

Where r is the perpendicular distance of the point from the wire and$\hat{n}$is the radial unit vector in the plane normal to the wire passing through the point.

● Electric field due to a Infinite thin plane sheet of uniform surface charge density σ

$$E = \frac{\sigma}{2\epsilon_0}\hat{n};$$

Where, $\hat{n}$ is a unit vector normal to the plane, outward on either side.

- Electric field due to a thin spherical shell of uniform surface charge density σ

$$E = 0 \quad (r < R)$$

$$E = \frac{q}{4\pi\epsilon_0 r^2}\,\mathrm{r}\,(r > R);$$

Where, 'r' is the distance of the point from the center of the shell, 'R' the radius of the shell and q is the total charge of the shell: $q = 4\pi R^2\sigma$.

The electric field outside the shell is as though the total charge is concentrated at the center. The same result is true for a solid conducting sphere of uniform volume charge density. **The field is zero at all points inside the shell.**